Acoustic Reflection Branch Functionals

A Causal Shell-Balance Framework for Prime-Exponent Phase Propagation

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Abstract

This paper reformulates the observed prime-exponent branch grammar as an acoustic reflection system. Earlier empirical work found that square-boundary residual classes $B_2(N)$ are partitioned by lifted prime-power phase channels p^k, with full-depth channels forming disjoint basins across tested prime prefixes. The deeper interpretation is that B_2 is only one member of a general reflection family B_i. For each exponent layer i, the driven shell functional

$$D_i(N, d_i) = d_i^i + i^2 N$$

interacts with the nearest i-power shell y_i^i. The reflected residual is

$$B_i(N, d_i) = d_i^i + i^2 N - y_i^i.$$

This is the closed-form branch functional. The identity

$$i^2 N = y_i^i - d_i^i + B_i(N, d_i)$$

shows that each branch state is an exact causal shell-balance reflection, not a statistical label.

The previously observed residue basins are reinterpreted as lifted phase shadows of these acoustic reflections. For source layer i, target layer j, and channel p^k, the cross-layer reflection basin is

$$S_{i\to j,p,k,r} = \{\, i^2 N \bmod p^k : B_j(N) = r \,\}.$$

The earlier experiments primarily studied $S_{i\to 2,p,k,r}$, showing that higher exponent reflections $B_3, B_5, B_7, \ldots$ recover phase information lost in the square reflection B_2. Tail closure is therefore reinterpreted as acoustic resolution: unresolved branch mass is target boundary-distance information below the current source-phase resolution, and later exponent layers act as higher shell modes that resolve it. A boundary-local causal sufficiency audit confirms that, over active channel domains, roughly 95%–96% of tested lifted channels are causally sufficient at full depth across the $10k$, $20k$, and $50k$ prime-prefix tests.

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1 Introduction

The original empirical investigation asked why square-boundary residuals form stable branch basins over lifted prime-power channels. The data showed that higher exponent layers recover branch

information lost by the square layer, and that full lifted phases p^k partition branch states without observed collision in the tested domains.

However, the deeper causal object is not merely the square residual $B_2(N)$. The correct object is the full branch-response family

$$B_i(N, d_i),$$

defined at every exponent layer i. The square branch B_2 is only the lowest visible reflection. Higher branches $B_3, B_5, B_7, \ldots$ are not auxiliary classifiers; they are higher acoustic shell reflections of the same integer N.

The closed form is therefore not first a finite residue basin. The closed form is the reflection equation:

$$B_i(N, d_i) = d_i^i + i^2 N - y_i(N, d_i)^i.$$

The lifted residue basins are phase shadows of this reflection.

2 Acoustic Interpretation

The system has a natural acoustic form:

Mathematical object	Acoustic interpretation
d_i^i	local displacement drive
$i^2 N$	scaled impedance/mass term
$D_i = d_i^i + i^2 N$	incident driven phase
y_i^i	nearest resonant shell
$B_i = D_i - y_i^i$	reflected residual / echo
$\rho_i = B_i/\Delta_i$	normalized reflection coefficient

Thus branch behavior is not random modular classification. It is a reflection response generated when a driven phase meets the nearest resonant shell.

3 Layer Reflection Functionals

Definition 1 (Driven Shell Functional)**.** *For $N \in \mathbb{Z}_{>0}$, exponent layer $i \geq 2$, and local displacement $d_i \in \mathbb{Z}_{\geq 0}$, define*

$$D_i(N, d_i) = d_i^i + i^2 N.$$

Definition 2 (Nearest i-Power Shell)**.** *Let*

$$y_i(N, d_i) = \arg\min_{q \in \mathbb{Z}_{\geq 0}} \left| D_i(N, d_i) - q^i \right|.$$

When two shells are equidistant, choose the lower shell by convention.

Definition 3 (Layer Reflection Functional)**.** *The i-layer branch reflection is*

$$B_i(N, d_i) = D_i(N, d_i) - y_i(N, d_i)^i.$$

Equivalently,

$$B_i(N, d_i) = d_i^i + i^2 N - y_i(N, d_i)^i.$$

Proposition 1 (Exact Reflection Balance)**.** *For every N, i, d_i,*

$$d_i^i + i^2 N - y_i(N, d_i)^i = B_i(N, d_i).$$

Equivalently,

$$i^2 N = y_i(N, d_i)^i - d_i^i + B_i(N, d_i).$$

Proof. This follows immediately from the definition of B_i:

$$B_i(N, d_i) = D_i(N, d_i) - y_i(N, d_i)^i$$

and

$$D_i(N, d_i) = d_i^i + i^2 N.$$

Substitution gives the first identity; rearrangement gives the second. □

4 Bounded Acoustic Reflection

The nearest-shell convention imposes a bound on the reflected residual.

Definition 4 (Adjacent Shell Gap)**.** *Let $y = y_i(N, d_i)$ be the selected nearest shell. Let y' be the adjacent competing shell:*

$$y' = \begin{cases} y + 1, & D_i(N, d_i) \geq y^i, \\ y - 1, & D_i(N, d_i) < y^i \text{ and } y > 0. \end{cases}$$

Define the local shell gap

$$\Delta_i(y, y') = |y'^i - y^i|.$$

When the lower shell y is used as reference, one may write

$$\Delta_i(y) = (y + 1)^i - y^i.$$

Definition 5 (Normalized Reflection Coefficient)**.** *Define*

$$\rho_i(N, d_i) = \frac{B_i(N, d_i)}{\Delta_i(y_i, y_i')}.$$

Proposition 2 (Bounded Reflection)**.** *Under nearest-shell selection,*

$$|B_i(N, d_i)| \leq \frac{1}{2} \Delta_i(y_i, y_i').$$

Consequently,

$$|\rho_i(N, d_i)| \leq \frac{1}{2}.$$

Proof. Since y_i^i is chosen as the nearest shell to D_i, the distance from D_i to y_i^i is no greater than the distance to the adjacent competing shell. The midpoint between adjacent shells is the maximum distance from a nearest shell before the competing shell becomes nearer. Therefore the signed residual magnitude is at most half the adjacent shell gap. □

This is the acoustic constraint: every branch residual is a bounded reflection inside the local shell cavity.

5 Prime-Exponent Shell Gap Phase

The shell gap itself carries arithmetic phase. For prime exponent i,

$$(y+1)^i - y^i = \sum_{\ell=0}^{i-1} \binom{i}{\ell} y^\ell.$$

If i is prime, then every intermediate binomial coefficient is divisible by i, so

$$(y+1)^i - y^i \equiv 1 \pmod{i}.$$

Proposition 3 (Prime Shell Gap Phase)**.** *If i is prime, then*

$$(y+1)^i - y^i \equiv 1 \pmod{i}.$$

Proof. By the binomial theorem,

$$(y+1)^i - y^i = \binom{i}{i-1} y^{i-1} + \binom{i}{i-2} y^{i-2} + \cdots + \binom{i}{1} y + 1.$$

For prime i, each coefficient $\binom{i}{\ell}$ for $1 \le \ell \le i-1$ is divisible by i. Therefore all nonconstant terms vanish modulo i, leaving

$$(y+1)^i - y^i \equiv 1 \pmod{i}.$$

□

This identity explains why diagonal exponent-carrier channels $i = p$ exhibit special low-depth structure: the shell itself contributes a forced $+1$ phase modulo the exponent prime.

6 Lifted Reflection Phase

The reflection balance gives a phase identity modulo every prime power.

Definition 6 (Lifted Reflection Phase)**.** *For source layer i, carrier prime p, and depth k, define*

$$\Phi_{i,p,k}(N, d_i) = i^2 N \bmod p^k.$$

By exact reflection balance,

$$i^2 N = y_i^i - d_i^i + B_i(N, d_i),$$

and therefore

$$\Phi_{i,p,k}(N, d_i) \equiv y_i(N, d_i)^i - d_i^i + B_i(N, d_i) \pmod{p^k}.$$

Thus the same phase can be viewed from two sides:

$$\underbrace{i^2 N}_{N\text{-side impedance phase}} \equiv \underbrace{y_i^i - d_i^i + B_i}_{\text{reflected shell-balance phase}} \pmod{p^k}.$$

7 Cross-Layer Reflection Basins

The empirical basin work studied how higher source layers recover lower-layer branch information. This is naturally a cross-layer reflection relation.

Definition 7 (Cross-Layer Reflection Basin)**.** *For source layer i, target layer j, carrier p^k, and target reflection residual r, define*

$$S_{i\to j,p,k,r} = \{\Phi_{i,p,k}(N, d_i) : B_j(N, d_j) = r\}.$$

Equivalently,

$$S_{i\to j,p,k,r} = \left\{i^2 N \bmod p^k : B_j(N, d_j) = r\right\}.$$

Using reflection balance,

$$S_{i\to j,p,k,r} = \left\{y_i^i - d_i^i + B_i(N, d_i) \bmod p^k : B_j(N, d_j) = r\right\}.$$

The earlier experiments primarily studied

$$S_{i\to 2,p,k,r},$$

where

$$i = 3, 5, 7, 11, \ldots.$$

Thus, the empirical result should be interpreted as:

> Higher exponent reflections B_i recover phase information lost in the square reflection B_2.

8 Causal Sufficiency and Boundary-Distance Scaling

The preceding framework defines each branch functional $B_i(N)$ as a canonical acoustic reflection:

$$B_i(N) = d_i(N)^i + i^2 N - y_i(N)^i.$$

Thus every branch value is generated by a deterministic boundary interaction. The remaining question is not statistical prediction, but causal sufficiency: when does a source-layer phase contain enough boundary-distance information to determine a target-layer reflection?

8.1 Target Reflection and Domain

Let $\mathcal{N} \subset \mathbb{Z}_{>0}$ be a finite analysis domain, and let $j \geq 2$ be a target layer. The target branch functional is

$$B_j : \mathcal{N} \to \mathbb{Z},$$

where

$$B_j(N) = d_j(N)^j + j^2 N - y_j(N)^j.$$

The target branch $B_j(N)$ measures the signed distance from the driven quantity

$$D_j(N, d_j) = d_j(N)^j + j^2 N$$

to the nearest j-power boundary shell. Thus $B_j(N)$ is a boundary-distance state, not a statistical label.

8.2 Source Reflection Phase

For a source layer i, carrier prime p, and lifted depth k, define the source phase

$$\Phi_{i,p,k}(N) = i^2 N \bmod p^k.$$

By exact shell balance,

$$\Phi_{i,p,k}(N) \equiv y_i(N)^i - d_i(N)^i + B_i(N) \pmod{p^k}.$$

Therefore $\Phi_{i,p,k}$ is the p^k-lifted projection of the source reflection balance.

8.3 Causal Sufficiency

Definition 8 (Causal Sufficiency)**.** *Let $\mathcal{N}$ be a domain, i a source layer, j a target layer, and p^k a lifted carrier channel. The source channel (i, p, k) is causally sufficient for the target reflection B_j over $\mathcal{N}$ if*

$$\Phi_{i,p,k}(N_1) = \Phi_{i,p,k}(N_2) \quad \Longrightarrow \quad B_j(N_1) = B_j(N_2)$$

for all $N_1, N_2 \in \mathcal{N}$.

Equivalently, (i, p, k) is causally sufficient when the lifted source phase has no unresolved target-boundary collisions over $\mathcal{N}$. This is not a statistical condition. It is a deterministic sufficiency condition. The source phase either preserves the target boundary-distance state or it does not.

8.4 Boundary-Distance Fibers

For a target reflection value r, define the causal fiber

$$S_{i\to j,p,k,r} = \{\Phi_{i,p,k}(N) : N \in \mathcal{N},\ B_j(N) = r\}.$$

If (i, p, k) is causally sufficient for B_j, then the fibers

$$\{S_{i\to j,p,k,r}\}_r$$

are pairwise disjoint. In other words,

$$r_1 \neq r_2 \quad \Longrightarrow \quad S_{i\to j,p,k,r_1} \cap S_{i\to j,p,k,r_2} = \varnothing.$$

Thus, causal sufficiency means that the source phase separates the target boundary-distance states.

8.5 Boundary Distance and Domain Scale

Causal sufficiency depends on the target layer j, the source layer i, the lifted depth p^k, and the domain $\mathcal{N}$.

To make the domain dependence explicit, define the target shell spacing near N by

$$\Delta_j(N) = y_{j,+}(N)^j - y_{j,-}(N)^j,$$

where $y_{j,-}^j$ and $y_{j,+}^j$ are the adjacent j-power shells straddling the driven target quantity $D_j(N, d_j)$.

The normalized target reflection distance is

$$\rho_j(N) = \frac{B_j(N)}{\Delta_j(N)}.$$

Under nearest-shell selection,

$$|\rho_j(N)| \le \frac{1}{2}.$$

The target boundary state is therefore controlled by where $D_j(N, d_j)$ falls inside its local shell interval. A source channel is causally sufficient when its lifted phase resolution is fine enough to distinguish all target boundary-distance states that occur inside $\mathcal{N}$.

8.6 Causal Resolution Scale

Define the causal resolution of the source channel as

$$R_{i,p,k} = p^k.$$

This is the number of distinguishable lifted phase states available in the source projection. A larger p^k gives a finer phase resolution.

Define the target boundary spread over $\mathcal{N}$ by

$$\mathcal{B}_j(\mathcal{N}) = \{B_j(N) : N \in \mathcal{N}\}.$$

The channel (i, p, k) must distinguish the target boundary-distance states in $\mathcal{B}_j(\mathcal{N})$. If two different target reflections collapse to the same source phase, then p^k is too shallow for that domain.

Definition 9 (Boundary Collision Set)**.** *The boundary collision set for source channel (i, p, k), target j, and domain $\mathcal{N}$ is*

$$\mathcal{C}_{i\to j,p,k}(\mathcal{N}) = \{\phi : \exists N_1, N_2 \in \mathcal{N} \text{ such that } \Phi_{i,p,k}(N_1) = \Phi_{i,p,k}(N_2) = \phi \text{ and } B_j(N_1) \neq B_j(N_2)\}.$$

Then causal sufficiency is exactly

$$\mathcal{C}_{i\to j,p,k}(\mathcal{N}) = \varnothing.$$

8.7 Minimal Causally Sufficient Depth

Definition 10 (Minimal Causally Sufficient Depth)**.** *For source layer i, target layer j, carrier prime p, and domain $\mathcal{N}$, define*

$$k_{\mathrm{cs}}(i \to j, p; \mathcal{N}) = \min\{k \ge 1 : \mathcal{C}_{i\to j,p,k}(\mathcal{N}) = \varnothing\},$$

when such a depth exists.

This is the first lifted depth at which the source reflection phase fully separates the target boundary-distance states over the domain.

8.8 Causal Sufficiency Scaling Law

Theorem 1 (Causal Sufficiency Scaling)**.** *Let*

$$\mathcal{N}_M \subseteq \mathcal{N}_{M'} \quad \textit{with} \quad M < M'.$$

If a source channel (i, p, k) is causally sufficient for target j over $\mathcal{N}_{M'}$, then it is causally sufficient over $\mathcal{N}_M$.

Equivalently, causal sufficiency over a larger domain implies causal sufficiency over every smaller subdomain.

Proof. If (i, p, k) is causally sufficient over $\mathcal{N}_{M'}$, then no two elements of $\mathcal{N}_{M'}$ with the same source phase have different target reflections. Since $\mathcal{N}_M \subseteq \mathcal{N}_{M'}$, the same is true for all pairs in $\mathcal{N}_M$. Therefore the channel is causally sufficient over $\mathcal{N}_M$. □

Corollary 1 (Nondecreasing Sufficiency Requirement)**.** *For increasing domains $\mathcal{N}_M \subseteq \mathcal{N}_{M'}$, the minimal causally sufficient depth satisfies*

$$k_{\mathrm{cs}}(i \to j, p; \mathcal{N}_M) \leq k_{\mathrm{cs}}(i \to j, p; \mathcal{N}_{M'})$$

whenever both quantities are defined for the same carrier p.

This is the formal scaling law. As the domain expands, new boundary-distance states may enter existing source-phase buckets. If those new states carry a different target reflection, the old lifted depth is no longer sufficient. The channel must then be refined by increasing k, adding another carrier p, or moving to another source layer i.

8.9 Tail as Causal Insufficiency

Define the unresolved tail at source channel (i, p, k) for target j by

$$T_{i\to j,p,k}(\mathcal{N}) = \{N \in \mathcal{N} : \Phi_{i,p,k}(N) \in \mathcal{C}_{i\to j,p,k}(\mathcal{N})\}.$$

Then

$$T_{i\to j,p,k}(\mathcal{N}) = \varnothing$$

if and only if (i, p, k) is causally sufficient for B_j over $\mathcal{N}$.

Thus, the tail is not error. The tail is exactly the set of rows whose target boundary-distance state is not yet separated by the current source reflection phase.

8.10 Layered Causal Closure

Let P be a set of carrier primes and let K_p be the lifted depth used for each p. Define the combined source phase signature

$$\Phi_{i,P,K}(N) = \left(i^2 N \bmod p^{K_p}\right)_{p\in P}.$$

Then (i, P, K) is causally sufficient for target j over $\mathcal{N}$ if

$$\Phi_{i,P,K}(N_1) = \Phi_{i,P,K}(N_2) \quad \Longrightarrow \quad B_j(N_1) = B_j(N_2).$$

This generalizes single-channel sufficiency. It also matches the empirical tail behavior: as the domain grows, closure may require multiple carriers, greater lifted depths, and later source layers.

Conjecture 1 (Layered Causal Closure)**.** *For prime-prefix domains $\mathcal{N}_M$, unresolved target reflection mass is closed by a finite layered source signature:*

$$\Phi_{\le i,P,K}(N) = \left(\ell^2 N \bmod p^{K_{\ell,p}}\right)_{\ell \le i,\ p \in P_\ell}.$$

For each tested scale M, there exists a finite set of source layers, carriers, and lifted depths such that

$$\Phi_{\le i,P,K}(N_1) = \Phi_{\le i,P,K}(N_2) \quad \Longrightarrow \quad B_j(N_1) = B_j(N_2)$$

over $\mathcal{N}_M$.

8.11 Interpretive Summary

A source reflection phase is causally sufficient when it preserves enough distance-to-boundary information to determine the target reflection. As the domain expands, more boundary-distance states appear. A phase projection that was sufficient at a smaller scale may become insufficient because two different target reflections now fall into the same source bucket.

The tail is exactly that unresolved collision set. Tail closure happens by increasing lifted depth, adding carrier channels, or moving to later exponent reflection layers.

Thus,

$$\boxed{\text{branching is always causally determined,}}$$

and

$$\boxed{\text{ambiguity is only insufficient boundary-distance resolution.}}$$

9 Branch Fibers and Disjointness

Definition 11 (Reflection Branch Map)**.** *Assume (i,p,k) is causally sufficient for target layer j. Define*

$$\beta_{i\to j,p,k} : \Phi_{i,p,k}(\mathcal{N}) \to \mathrm{Image}(B_j)$$

by

$$\beta_{i\to j,p,k}(\Phi_{i,p,k}(N,d_i)) = B_j(N,d_j).$$

Definition 12 (Reflection Branch Fiber)**.** *For target residual r, define*

$$S_{i\to j,p,k,r} = \beta^{-1}_{i\to j,p,k}(r).$$

Theorem 2 (Disjoint Reflection Fibers)**.** *If (i,p,k) is causally sufficient for B_j, then the fibers*

$$S_{i\to j,p,k,r}$$

are pairwise disjoint:

$$r_1 \neq r_2 \quad \Longrightarrow \quad S_{i\to j,p,k,r_1} \cap S_{i\to j,p,k,r_2} = \varnothing.$$

Proof. Fibers of a function are disjoint. If a phase ϕ belonged to two distinct fibers, then $\beta_{i\to j,p,k}(\phi)$ would equal two different values, contradicting that β is a function. □

10 Tail Closure as Acoustic Resolution

The unresolved tail is not noise. It is unresolved reflected phase.

Definition 13 (Unresolved Reflection Tail)**.** *For source layer i, target layer j, and channel p^k, define $T_{i\to j,p,k}(\mathcal{N})$ as the set of $N \in \mathcal{N}$ whose target branch $B_j(N)$ is not uniquely resolved by $\Phi_{i,p,k}(N)$.*

Increasing k refines the p-adic phase resolution:

$$\mathbb{Z}/p^{k+1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}.$$

Thus higher k splits ambiguous phase buckets. Moving to later prime exponent layers $i = 5, 7, 11, \ldots$ introduces new shell modes, analogous to higher acoustic harmonics.

Therefore:

tail closure = resolving reflected phase by higher harmonic shells.

11 Boundary-Local Causal Sufficiency Audit

The causal sufficiency framework distinguishes between a global prefix domain and the active boundary domain of a channel. A single reflection channel is not required to resolve every integer in the full prime prefix. Acoustically, a mode resolves the portion of the signal that couples into that mode.

The first global audit tested whether each safe-rule channel (i, p, k) determined B_2 over the entire prime prefix $\mathcal{N}_M$. This test was intentionally too strong. No single channel was causally sufficient over the entire prefix. The best global channels still left large unresolved tails. This does not contradict the reflection law; it shows that the correct causal domain is boundary-local rather than prefix-global.

11.1 Active Boundary Domain

For each lifted channel (i, p, k) and prefix scale M, define the active boundary domain

$$\mathcal{D}_{i,p,k}(M) = \{N \in \mathcal{N}_M : N \text{ appears in the active boundary records of } (i, p, k)\}.$$

This domain is the set of rows that actually couple into the channel's reflection mode. Causal sufficiency for a channel should therefore be tested over

$$\mathcal{D}_{i,p,k}(M),$$

not over the entire prefix $\mathcal{N}_M$.

11.2 Boundary-Local Sufficiency Condition

The boundary-local causal sufficiency condition is

$$\Phi_{i,p,k}(N_1) = \Phi_{i,p,k}(N_2) \quad \Longrightarrow \quad B_2(N_1) = B_2(N_2)$$

for all

$$N_1, N_2 \in \mathcal{D}_{i,p,k}(M).$$

Equivalently, the boundary-local collision set

$$\mathcal{C}_{i\to 2,p,k}(\mathcal{D}_{i,p,k}(M))$$

is empty.

This is the correct acoustic statement: a source reflection phase determines the target boundary-distance state over the part of the domain that is active in that source reflection mode.

11.3 Audit Results

The boundary-local audit tested the active domains extracted from the full-value basin records at the $10k$, $20k$, and $50k$ prime prefixes. The results were:

Prefix	Channels	Sufficient	Fraction	Total active rows over channels
$10k$	42	40	95.24%	2090
$20k$	56	54	96.43%	6092
$50k$	63	60	95.24%	16651

Thus, over active boundary domains, roughly 95%–96% of tested channels were causally sufficient at full lifted depth.

The strongest sufficient channels at $50k$ were the same trunk channels that appeared throughout the earlier basin analysis:

Source layer i	p^k	k_{cs}	active rows
3	11^4	4	2009
3	23^3	3	1169
3	5^6	6	1165
3	19^3	3	870
3	7^5	5	791
5	11^4	4	514

This confirms that the boundary-local causal sufficiency audit is not discovering a separate structure. It formalizes the same lifted reflection channels found in the union-basin and shell-balance audits.

11.4 Remaining Boundary-Local Tails

The remaining unresolved channels are few and informative. At $50k$, the unsolved active-domain tails were:

i	p^k	tail	active rows
5	5^6	359	603
3	3^8	176	357
3	3^7	2	35

The 3-adic channels remain mildly problematic across scales, while the diagonal $i = 5, p = 5, k = 6$ channel has a larger unresolved tail at $50k$. This is consistent with the earlier closed-form cylinder result: the diagonal channel compresses strongly in a diagnostic phase involving the target branch, but raw source phase $i^2N \bmod p^k$ does not yet fully resolve all active target states.

This suggests that the remaining tails require source-side acoustic terms beyond the raw lifted phase, such as

$$y_i^i - d_i^i + B_i(N), \qquad B_i(N), \qquad \rho_i(N), \qquad \Delta_i(N),$$

or combinations of these terms.

11.5 Corrected Causal Law

The corrected causal law is boundary-local:

> A reflection channel is causally sufficient over its active boundary domain, not necessarily over the entire prefix.

Equivalently,

> $\Phi_{i,p,k}(N)$ resolves $B_2(N)$ for $N \in \mathcal{D}_{i,p,k}(M)$
>
> when the boundary-local collision set is empty.

The global-prefix audit failed because it asked a channel to resolve integers outside its acoustic coupling domain. The boundary-local audit succeeds for nearly all tested channels and identifies the remaining tails as genuine unresolved boundary-distance collisions.

12 Empirical Findings Reinterpreted

12.1 Prime-Exponent Cascade

The empirical cascade showed that the square layer alone was insufficient, but higher layers progressively closed the tail. At $50k$, the closure sequence was:

$$\begin{aligned} i = 2 &\to 20.716\%, \\ i = 3 &\to 81.696\%, \\ i = 5 &\to 96.494\%, \\ i = 7 &\to 99.524\%, \\ i = 11 &\to 100\%. \end{aligned}$$

In acoustic terms, the B_2 reflection is a low-resolution reflection measurement. Higher reflections $B_3, B_5, B_7, \ldots$ recover phase content lost in the square shell.

12.2 Lifted Prime-Power Channels

The dominant closure family was N_{cmod,p^k}, now interpreted as a lifted reflection phase:

$$\Phi_{i,p,k}(N) = i^2 N \bmod p^k.$$

The major observed carriers included:

$$11^4, \quad 5^6, \quad 23^3, \quad 7^5, \quad 19^3, \quad 17^3.$$

12.3 Union Basin Disjointness

Full-value union tests found:

$$\begin{aligned} \text{Channels tested} &= 80, \\ \text{Disjoint channels} &= 80, \\ \text{Collision rows} &= 0, \\ \text{Safe rules} &= 576. \end{aligned}$$

In reflection language, these are disjoint cross-layer reflection fibers.

12.4 Closed-Form Cylinder Mining

The closed-form cylinder miner tested phase coordinates and found that diagonal exponent-carrier channels compress especially well. The most striking case was

$$i = 5, \quad p = 5, \quad k = 6,$$

where 608 values and 14 residual classes compressed to 14 pure cylinders at average depth 2.

This is explained by the diagonal identity. If $i = p$, then

$$i^2 N \equiv 0 \pmod{i^2}.$$

Thus for the diagnostic phase

$$\phi = i^2 N - B_j(N),$$

one has

$$\phi \equiv -B_j(N) \pmod{i^2}.$$

So every target residual r lies in the cylinder

$$\phi \equiv -r \pmod{i^2}.$$

Proposition 4 (Diagonal Exponent-Carrier Cylinder)**.** *Let $i = p$ be prime and define*

$$\phi(N) = i^2 N - B_j(N, d_j).$$

Then

$$\phi(N) \equiv -B_j(N, d_j) \pmod{i^2}.$$

Consequently, if $B_j(N, d_j) = r$, then

$$\phi(N) \equiv -r \pmod{i^2}.$$

Proof. Since $i^2 N$ is divisible by i^2,

$$i^2 N \equiv 0 \pmod{i^2}.$$

Therefore

$$i^2 N - B_j(N, d_j) \equiv -B_j(N, d_j) \pmod{i^2}.$$

If $B_j(N, d_j) = r$, the result follows. □

This is diagnostic rather than predictive, because it uses the target branch B_j in the phase expression. But it confirms that diagonal channels have a forced low-depth reflection signature.

13 Predictive Versus Diagnostic Closed Forms

There are two distinct kinds of closed form.

13.1 Diagnostic Closed Forms

Diagnostic forms may use the target branch B_j. They verify consistency and expose acoustic structure. For example:

$$i^2 N - B_j(N) \equiv -B_j(N) \pmod{i^2}$$

when $i = p$.

13.2 Predictive Closed Forms

Predictive forms must not use B_j on the left-hand side. They must use only source-layer reflection terms:

$$i^2N, \quad y_i^i - d_i^i, \quad B_i(N), \quad \Delta_i, \quad \rho_i.$$

A predictive cross-layer law would have the form

$$B_j(N) = r \quad \Longleftrightarrow \quad F_i(N, d_i, y_i, B_i) \equiv a_r \pmod{p^k},$$

where F_i contains no B_j.

The empirical work has established the existence of clean cross-layer fibers. The next mathematical task is to derive the predictive generator F_i from the acoustic reflection equation.

14 Formal Summary

The general closed form is the layer reflection:

$$\boxed{B_i(N, d_i) = d_i^i + i^2N - y_i(N, d_i)^i.}$$

The propagation balance is:

$$\boxed{i^2N = y_i^i - d_i^i + B_i(N, d_i).}$$

The lifted reflection phase is:

$$\boxed{\Phi_{i,p,k}(N, d_i) = i^2N \bmod p^k.}$$

The cross-layer basin is:

$$\boxed{S_{i\to j,p,k,r} = \{\, i^2N \bmod p^k : B_j(N, d_j) = r \,\}.}$$

The observed B_2 basin grammar is therefore the special case

$$\boxed{S_{i\to 2,p,k,r}.}$$

The tail is unresolved reflection phase:

$$\boxed{T_{i\to j,p,k} = \{N : \Phi_{i,p,k}(N) \text{ does not yet resolve } B_j(N)\}.}$$

And tail closure is higher-shell acoustic resolution:

$$\boxed{\text{tail closure} = \text{phase resolution by lifted depth and higher exponent shell modes.}}$$

15 Limitations and Next Work

This paper establishes the reflection functional and reframes the observed basins as lifted phase shadows of acoustic shell balance. Several tasks remain:

1. Formalize the displacement selection convention d_i for each layer.

2. Derive predictive cross-layer generator functions F_i that do not contain the target branch B_j.

3. Extend full-value tests to larger prime prefixes.

4. Test whether cross-layer reflection fibers remain disjoint under scale growth, especially over active boundary domains $\mathcal{D}_{i,p,k}(M)$.

5. Treat composites separately as multi-reflection or interference states rather than forcing prime-state reflection grammar onto them.

16 Conclusion

The branch grammar is acoustic reflection. The square residual B_2 is not a special isolated object; it is the lowest layer of a family of reflection responses

$$B_i(N, d_i) = d_i^i + i^2 N - y_i^i.$$

Higher exponent layers are higher shell modes. They recover reflected phase information lost in the square shell. Lifted prime-power basins are not the cause of the branch structure; they are phase shadows of the underlying reflection equation.

The boundary-local causal sufficiency audit confirms the correct domain for this law. A single channel is not expected to resolve every integer in a prime prefix. It resolves the rows that couple into its active boundary mode. Over those active domains, the tested channels are overwhelmingly causally sufficient at full lifted depth, with remaining tails identifying the few unresolved boundary-distance collisions that require additional source-side acoustic terms.

The central causal law is:

$$d_i^i + i^2 N - y_i^i = B_i(N, d_i).$$

The central propagation law is:

$$i^2 N = y_i^i - d_i^i + B_i(N, d_i).$$

And the empirical basin law is the cross-layer shadow:

$$B_j(N) = r \quad \Longleftrightarrow \quad i^2 N \bmod p^k \in S_{i \to j,p,k,r}$$

whenever the source channel is causally sufficient.

Thus, branch propagation is not a clever classification strategy. It is a causal acoustic reflection process.